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BRANCHING PARTICLE REPRESENTATION OF A CLASS OF SEMILINEAR EQUATIONS

José Alfredo López-Mimbela

We review several probabilistic techniques that were developed in a series of papers to study blowup properties of positive (mild) solutions of semilinear equations of the form $\partial u(t,x)/\partial t = Au(t,x)+u^{\beta}(t,x)$, $u(0,x)=f(x)$, where A is the generator of a strong Markov process in a locally compact space S, $\beta > 1$ is an integer, and $f : S \to [0,+\infty)$ is bounded and measurable. The emphasis is on probabilistic representations of positive solutions, and on qualitative properties of solutions.

1. Introduction

This paper constitutes a report on probabilistic methods that were developed in [6], [7], [8] and [9] to study blow-up properties of semilinear equations of the prototype

$$\frac{\partial u_t}{\partial t} = Au_t + Vu_t^{\beta}, \quad u_0 = f, \tag{1}$$

where A denotes the infinitesimal generator of a strong Markov process in a state space S, $V > 0$ and $\beta > 1$ are constants, and the initial condition $f : S \to [0,+\infty)$ is bounded and measurable. Reaction-diffusion equations of the form (1) are related to important questions of qualitative nature in many fields of application,

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and have been studied intensively in the last three decades because of the rich mathematical structure associated with their qualitative behavior. See [5], [12], [14] or [15] for surveys.

Under appropriate conditions on S there exists an extended real number $T_f > 0$ such that (1) has a unique solution u on $S \times [0, T_f)$ which is bounded on $S \times [0, T]$ for any $0 < T < T_f$, and if $T_f < \infty$, then $\|u_t\|_\infty \to +\infty$ as $t \uparrow T_f$. When $T_f = +\infty$ we say that u is a global solution, and when $T_f < +\infty$ we say that u blows up in finite time, or that u is nonglobal.

In his pioneering paper [3] Fujita showed, initially for the case $S = \mathbb{R}^d$ (where $\mathbb{R}^d$ is d-dimensional Euclidean space), $A = \Delta := \sum_{i=1}^d \partial^2/\partial x_i^2$ and $V = 1$, that the spatial dimension d and the exponent β in the nonlinearity play a crucial role in the asymptotic behavior of positive solutions of (1). His results state that if $d(\beta - 1)/2 > 1$, then Equation (1) admits both global and nonglobal positive solutions, and that if $0 < d(\beta - 1)/2 < 1$, then (1) has no nontrivial global positive solution.

The probabilistic counterpart to Fujita's results appeared soon after the publication of [3]. In [13] Nagasawa and Sirao expounded a probabilistic method, based on the theory of semigroups with the branching property that was developed in [4], that allowed them to re-discover Fujita's results in the case of integral exponents $\beta \geq 2$, and a generator A of a Markovian migration in a compact space. Later, the present author introduced in [6] a probabilistic representation of mild solutions of (1), and extended the results on existence of global solutions in [13] to certain systems of semilinear equations. The blowup of systems of equations was treated probabilistically later on, in the papers [8] and [9]. By combining analytic and probabilistic tools, a Dirichlet boundary value problem related to (1) was studied in [7].

A common characteristic of the probabilistic approaches developed in these papers (with the exception of [7]) is the use of Markov branching processes to represent positive mild solutions of (1) as expectation functionals. This feature restricts their scope to semilinear equations with integer exponents $\beta \geq 2$ in the nonlinearities. However, they provide a way by which one can explain in a transparent and intuitive probabilistic manner why blowup occurs under certain constellations of parameters. Moreover, by considering multitype branching systems, one can easily extend the analysis to systems of equations.

Our purpose in this work is to review the main results in [6, 7, 8] and [13] in a reasonably unified context. In Section 2 we briefly recall the construction of a Markov branching process introduced by Ikeda, Nagasawa and Watanabe. Afterward, in Section 3, we give a probabilistic representation of mild solutions of (1) which is used through sections 4 and 5 to derive sufficient conditions for blowup and for existence of global solutions. Section 6 deals with systems of semilinear equations. Section 7 constitutes an attempt to interpret blowup for the Dirichlet boundary value problem in terms of our branching process representations.

2. Markov Branching processes

Let us describe the Markov branching processes by means of which we are going to represent solutions of (1). We refer to [4] for a complete presentation of this topic.

Let S be a Hausdorff, locally compact, second countable topological space. Let us denote by $\mathcal{N}_{\mathsf{f}}(S)$ the space of finite counting measures on S, endowed with the topology of vague convergence. We write $\mathrm{supp}(\mu)$ for the support of $\mu \in \mathcal{N}_{\mathsf{f}}(S)$. The space of bounded, Borel measurable functions $f : S \to \mathbb{R}_+$ (where $\mathbb{R}_+ := [0,\infty)$) will be denoted by $\mathcal{B}(S)$. We also write $\mathcal{B}(E)$ for the Borel σ-algebra in a topological space E.

To each $f \in \mathcal{B}(S)$ we associate a new measurable function $\hat{f} : \mathcal{N}_{\mathsf{f}}(S) \to \mathbb{R}_+$, defined by

$$\hat{f}(\mu) = \prod_{x \in \mathrm{supp}(\mu)} f(x), \;\; \mu \in \mathcal{N}_{\mathsf{f}}(S).$$

Let $\pi(x, B)$ be a function defined on $S \times \mathcal{B}(\mathcal{N}_{\mathsf{f}}(S))$ having the properties:

$$\text{(2)} \qquad \pi(\cdot, B) \quad \text{is } \mathcal{B}(S)\text{-measurable for each } B \in \mathcal{B}(\mathcal{N}_{\mathsf{f}}(S))\,,$$

$$\text{(3)} \qquad \pi(x, \cdot) \quad \text{is a probability measure on } \mathcal{B}(\mathcal{N}_{\mathsf{f}}(S)) \text{ for any } x \in S\,,$$

$$\text{(4)} \qquad \pi(x, \mathcal{N}_{[1]}) \;\; = 0 \text{ for each } x \in S,$$

where $\mathcal{N}_{[n]} \subset \mathcal{N}_{\mathsf{f}}(S)$ consists of the measures having exactly n atoms, $n = 1, 2, \ldots$.

Given a strong Markov process $W =: \{W_t,\ t \geq 0\}$ with values in S and a bounded, measurable function $V : S \to (0,\infty)$, there exists a unique Markov process $X := \{X_t,\ t \geq 0\}$ with state space $\mathcal{N}_{\mathsf{f}}(S)$ whose paths are right continuous and have limits on the left at any point $t > 0$, and such that

$$\text{(5)} \qquad \mathbb{E}_\mu[\hat{f}(X_t)] = \prod_{x \in \mathrm{supp}(\mu)} \mathbb{E}_x[\hat{f}(X_t)],\;\; f \in \mathcal{B}(S),\;\; \mu \in \mathcal{N}_{\mathsf{f}}(S),$$

$$\text{(6)} \qquad \{X_t,\ t < T,\} \stackrel{\mathrm{d}}{=} \{Y_t,\ t < T\},$$

($\stackrel{\mathrm{d}}{=}$ meaning equality in distribution), where $Y := \{Y_t,\ t \geq 0\}$ is a Markov process with state space $S \cup \{\dagger\}$ ($\dagger$ being an extra point), whose lifetime is T, has $\dagger$ as its terminal point, and obeys

$$\mathbb{P}_x[\,Y_t \,\in\, B\,] \,=\, \mathbb{E}_x[e^{-\int_0^t V(W_s)\,ds}, W_t \in B],\ x \in S.$$

Moreover, for any $\lambda \geq 0$,

$$(7)\quad \mathbb{E}_x[\,e^{-\lambda T},\ X_T \,\in\, B \,|\, X_{T-}\,] \,=\, \mathbb{E}_x[\,e^{-\lambda T} \,|\, X_{T-}\,]\,\pi(X_{T-}, B)\ \text{ a.s. on } \{T < \infty\}$$

for any $B \in \mathcal{B}(\mathcal{N}_{\mathsf{f}}(S))$ and $x \in S$. Here $\mathbb{E}_\mu$ and $\mathbb{P}_\mu$ denote, respectively, conditional expectation and probability given that $X_0 = \mu$. In case of $\mu = \delta_x$ we write simply $\mathbb{E}_x$ and $\mathbb{P}_x$.

The process X is termed a "Markov branching process" [4]. Property (5) is usually referred to as the branching property. The process $\{Y_t\,,\, t \geq 0\}$ in (6) is the non-branching part of X, and the function π satisfying (2)-(4) and (7) is the branching law of X.

The process X starting in $X_0 = \delta_x$ describes the evolution of a population in S whose space-time behavior can be explained intuitively in the following way. Initially (i.e., at time $t = 0$) there is an individual at position x that migrates following the process W. After an exponentially distributed lifetime of parameter V it branches, originating an offspring with distribution π. The new particles evolve independently following the same rules. The random measure X_t represents the population configuration at time $t \geq 0$.

3. Representation of solutions

Let us consider the branching population defined in the previous section. In order to represent positive mild solutions of

$$(8)\qquad \frac{\partial u_t(x)}{\partial t} = Au_t(x) + Vu_t^\beta(x),\quad t > 0,\quad u_0(x) = f(x),\quad x \in S,$$

we take a constant function $V(x) \equiv V > 0$, a conservative Markov process $\{W_t,\ t \geq 0\}$ with values in S having infinitesimal generator A and semigroup $\{T_t,\ t \geq 0\}$ given by

$$T_t f(x) := \mathbb{E}_x[f(W_t)] = \int f(y)\, q_t(x, dy),\ t \geq 0,\ x \in S,\ f \in \mathcal{B}_b(S),$$

where $\{q_t(x, dy),\ t > 0\}$ is a family of transition kernels of $\{W_t\,,\, t \geq 0\}$. The branching law is given by $\pi(x, d\mu) = \delta_{\beta\delta_x}(d\mu)$, $x \in S$, where $\beta \geq 2$ is an integral constant. For $f \in \mathcal{B}(S)$ we define

$$w_t(\mu) \,=\, \mathbb{E}_\mu\left[e^{S_t}\hat{f}(X_t)\right],\ \ \mu \in \mathcal{N}_{\mathsf{f}}(S),\ \ t \geq 0,$$

where S_t denotes the "weighted occupation time"

$$S_t = V \int_0^t \int_S X_s(dx)\, ds = V \int_0^t N_s\, ds, \quad t \geq 0,$$

N_t being defined as the number of individuals in the population at time t. When $V = 1$, S_t coincides with the time length of the ancestor's offspring tree up to time t.

Theorem 1. *Let*

$$u_t(x) := \mathbb{E}_x \left[e^{S_t} \hat{f}(X_t) \right], \quad t \geq 0, \quad x \in S. \tag{9}$$

Then u_t is the mild solution of the initial value problem

$$\begin{aligned} \frac{\partial u_t(x)}{\partial t} &= Au_t(x) + V u_t^\beta(x), \quad t > 0, \\ u_0 &= f, \quad f \in \mathcal{B}(S). \end{aligned} \tag{10}$$

P r o o f. Let $X_0 = \mu = \sum_{i=1}^n \delta_{x_i}$ be the initial populaton of the branching system. Then the first branching time σ has exponential distribution of parameter nV. The law of total probability gives

$$w_t(\mu) = e^{-nVt} \mathbb{E}_\mu \left[e^{S_t}\, \hat{f}(X_t) \,\middle|\, \sigma > t \right] + \int_0^t nV e^{-nVs} \mathbb{E}_\mu \left[e^{S_t} \hat{f}(X_t) \,\middle|\, \sigma = s \right] ds.$$

Given that $\sigma \geq s$, the evolution of the population up to time s follows a stochastic translation originated by the motions of particles, hence the occupation time S_s equals $\int_0^s nV dr = nVs$. Noting that any given particle performs the first branching with probability $1/n$, it follows that

$$\begin{aligned} w_t(\mu) = {} & e^{-nVt} e^{\int_0^t nV\, dr} \prod_{i=1}^n T_t f(x_i) + \\ & + V \sum_{i=1}^n \int_0^t e^{-nVs} e^{\int_0^s nV\, dr} T_s \left(\int w_{t-s}(\nu) \pi^{(\cdot)}(d\nu) \right) (x_i) \prod_{l=1, l \neq i}^n T_t w_{t-s}(x_l)\, ds, \end{aligned}$$

where $\pi^{(z)}(d\nu) = \delta_{\beta\delta_z}(d\nu)$, $z \in S$. Therefore,

$$w_t(\mu) = \prod_{i=1}^n T_t f(x_i) + V \sum_{i=1}^n \int_0^t T_s \left(\int w_{t-s}(\nu) \pi^{(\cdot)}(d\nu) \right) (x_i) \prod_{l=1, l \neq i}^n T_t w_{t-s}(x_l)\, ds.$$

Putting $\mu = \delta_x$ and $u_t(x) := w_t(\delta_x)$ yields

$$u_t(x) = T_t f(x) + V \int_0^t T_s \left(u_{t-s}^\beta \right)(x)\, ds, \quad x \in S, \quad t \geq 0,$$

which is the integral form of (10). $\square$

4. Existence of global solutions

For any measurable function $v : \mathcal{N}_{\mathsf{f}}(S) \to \mathbb{R}_+$ we define the kernel $\mathbf{\Psi}$ by

$$\int_{\mathcal{N}_{\mathsf{f}}(S)} v(\nu)\, \mathbf{\Psi}(\mu, ds\, d\nu) = V \sum_{i=1}^{n} T_s \left(\int v(\nu)\, \pi^{(\cdot)}(d\nu) \right)(x_i) \prod_{\substack{l=1\\ l\neq i}}^{n} T_s v(x_l)\, ds,$$

$$\mu = \sum_{i=1}^{n} \delta_{x_i} \in \mathcal{N}_{\mathsf{f}}(S).$$

In terms of our branching model, $\mathbf{\Psi}(\mu, ds\, d\nu)$ represents a dynamics in which the "initial population" μ is transformed into a new one, ν, by a branching at time s of the ith particle, $i = 1, \ldots, n$. The remaining particles δ_{x_l}, $l \neq i$, do not branch, but develop independent motions according to the semigroup $\{T_r,\, r \geq 0\}$.

Let $u_t(x)$ be the function defined by (9). Then $\hat{u}_t = w_t$, $t \geq 0$, and, for any $f \in \mathcal{B}(S)$,

$$\hat{u}_t(\mu) \;=\; \widehat{T_t f}(\mu) + \int_0^t \int_{\mathcal{N}_{\mathsf{f}}(\mathbb{R}^d)} \hat{u}_{t-s}(\nu) \mathbf{\Psi}(\mu, ds\, d\nu).$$

Plugging the expression for $\hat{u}_t$ into the integrals of the right-hand side of the above equality renders

$$\hat{u}_t(\mu) \;=\; \sum_{k=0}^{\infty} u_k(t,\mu), \;\; \mu \,\in\, \mathcal{N}_{\mathsf{f}}(S), \;\; t \,\geq\, 0, \tag{11}$$

where $u_0(t,\mu) = \widehat{T_t f}(\mu)$ and

$$u_{k+1}(t,\mu) \;\;=\;\; \int_0^t \int_{\mathcal{N}_{\mathsf{f}}(\mathbb{R}^d)} u_k(t-s,\nu)\, \mathbf{\Psi}(\mu, ds\, d\nu), \quad k = 0, 1, \ldots.$$

Notice that $u_k(t,\mu) = \mathbb{E}_\mu \left[e^{S_t} \hat{f}(X_t); \kappa_t = k \right]$, $k = 0, 1, \ldots$, where κ_t denotes the number of branchings occurred up to time t. Hence, if $\mu = \sum_{i=1}^n \delta_{x_i}$,

$$\begin{aligned} u_1(t,\mu) \;&=\; V \int_0^t \sum_{i=1}^{n} T_s \left(\int \widehat{T_{t-s} f}(\nu) \pi^{(\cdot)}(d\nu) \right)(x_i) \prod_{\substack{l=1\\ l\neq i}}^{n} T_s \widehat{T_{t-s} f}(x_l) \\ &\leq\; V n \prod_{l=1}^{n} T_t f(x_l) \int_0^t \left(\sup_{z \in S} T_s f(z) \right)^{\beta - 1} ds, \end{aligned}$$

where we have used that $\widehat{T_r f}(\delta_z) = T_r f(z)$, $z \in S$, $t \geq 0$. Therefore,

$$u_1(t,\mu) \quad \leq \quad Vn\widehat{T_t f}(\mu) \int_0^t \left(\sup_{z\in S} T_s f(z) \right)^{\beta-1} ds, \quad \mu = \sum_{i=1}^n \delta_{x_i}, \quad t \geq 0.$$

Using induction, it can easily be verified that for any $k \geq 1$, $\mu = \sum_{i=1}^n \delta_{x_i}$ and $t \geq 0$,

$$u_k(t,\mu) \leq \frac{V^k}{k!} \prod_{i=0}^{k-1} (n + i(\beta-1)) \left[\int_0^t \left(\sup_{z\in S} T_s f(z) \right)^{\beta-1} ds \right]^k \widehat{T_t f}(\mu). \tag{12}$$

This and (11) yield the following theorem.

Theorem 2. *The mild solution $u_t(x)$ of Equation (10) satisfies*

$$u_t(x) \; \leq \; T_t f(x) \left(1 + \sum_{k=1}^\infty v_k(t) \right), \;\; x \in S, \;\; t \geq 0,$$

where

$$v_k(t) \;\; = \;\; \frac{\prod_{i=0}^{k-1}(1 + i(\beta-1))}{k!} \left[V \int_0^t \left(\sup_{z\in S} T_s f(z) \right)^{\beta-1} ds \right]^k.$$

In particular, for any $f \in \mathcal{B}(S)$ satisfying

$$(\beta - 1)V \int_0^\infty \left(\sup_{z\in S} T_s f(z) \right)^{\beta-1} ds < 1, \tag{13}$$

the corresponding solution of (10) is global, and

$$u_t(x) \leq \text{Const.}\, T_t f(x), \;\; x \in S, \;\; t \geq 0.$$

P r o o f. The assertion follows from (12) and the fact that $\sum_{k=1}^\infty v_k(t) < \infty$ uniformly in $t \geq 0$ due to (13). □

5. Finite time blowup

Lemma 1. *Let $K > 0$, and let*

$$\tilde{w}_t(\mu) := \mathbb{E}_\mu \left[e^{S_t} K^{N_t} \right], \quad t \geq 0,$$

where $\mu = \sum_{i=1}^n \delta_{x_i} \in \mathcal{N}_{\mathsf{f}}(S)$. *Then,*

$$\tilde{w}_t(\mu) = K^n \left[1 + \sum_{k=1}^{\infty} \left(\prod_{i=1}^{k} (n + (i-1)(\beta-1))\right) \frac{\left(VtK^{\beta-1}\right)^k}{k!}\right], \ n = 1, 2, \dots .$$

In particular, $\tilde{w}_t(\delta_x) = \infty$ *for any* $x \in S$ *provided that* $K \geq \left(\frac{1}{Vt(\beta-1)}\right)^{\frac{1}{\beta-1}}$.

P r o o f. Notice that both S_t and N_t are independent of the space variable. Hence, if $\mu = \sum_{i=1}^n \delta_{x_i}$, then

$$\tilde{w}_t(\mu) = \tilde{u}_t(n) := \mathbb{E}\left[e^{S_t^{(n)}} K^{N_t^{(n)}}\right],$$

where $S_t^{(n)}$ and $N_t^{(n)}$ denote, respectively, the quantities S_t and N_t corresponding to an initial configuration consisting of $n \geq 1$ particles. As before, conditioning on the first branching time we obtain

$$\mathbb{E}\left[e^{S_t^{(n)}} K^{N_t^{(n)}}\right] = e^{-nVt} e^{nVt} K^n + V \int_0^t ds\, e^{-nVs} e^{nVs} \sum_{i=1}^{n} \mathbb{E}\left[e^{S_{t-s}^{(n+\beta-1)}} K^{N_{t-s}^{(n+\beta-1)}}\right],$$

namely,

$$\tilde{u}_t(n) = K^n + nV \int_0^t \tilde{u}_s(n+\beta-1)\, ds, \quad n = 1, 2, \dots . \tag{14}$$

By iteration of (14) we find that $\tilde{u}_t(n)$ admits the series expansion

$$\tilde{u}_t(n) = u_t^{(0)}(n) + u_t^{(1)}(n) + \cdots, \tag{15}$$

where $u_t^{(0)}(n) = K^n$ and $u_t^{(k+1)}(n) = nV \int_0^t u_s^{(k)}(n+\beta-1)\, ds$, $n \geq 1$. Therefore,

$$\tilde{u}_t(n) = K^n \left[1 + \sum_{k=1}^{\infty} \left(\prod_{i=1}^{k} (n + (i-1)(\beta-1))\right) \frac{\left(VtK^{\beta-1}\right)^k}{k!}\right], \ n = 1, 2, \dots .$$

Taking $n = 1$ in the above expression and using that $\beta \geq 2$, we obtain

$$\begin{aligned} \mathbb{E}\left[e^{S_t} K^{N_t}\right] &\geq K\left[1 + \sum_{k=1}^{\infty} (\beta-1)^{k-1} (k-1)! \frac{\left(VtK^{\beta-1}\right)^k}{k!}\right] \\ &= K\left[1 + VtK^{\beta-1} \sum_{k=1}^{\infty} \frac{\left(Vt(\beta-1)K^{\beta-1}\right)^{k-1}}{k}\right]. \end{aligned}$$

The right-hand side of the last equality is infinite for $K \geq \left(\frac{1}{Vt(\beta-1)}\right)^{\frac{1}{\beta-1}}$. $\square$

Thus, in the absence of motion, Eq. (10) always blows up in finite time if $f \geq 0$ and $f \not\equiv 0$. This follows by a direct verification, or from Lemma 1 and the fact that

$$h_t \equiv \mathbb{E}\left[e^{S_t} f^{N_t}\right] = f + V \int_0^t h_r^\beta \, dr, \quad t \geq 0,$$

is the mild solution of

$$\frac{\partial h_t}{\partial t} = V h_t^\beta, \quad h_0 = f,$$

which blows up at $t_0 = V(\beta-1)^{-1} K^{1-\beta}$ provided $K := f(x) > 0$.

Lemma 2. *Let $\mathcal{T} \equiv \{\mathcal{T}_t,\ t \geq 0\}$ be the offspring tree of an ancestor δ_x, and let $f \in \mathcal{B}(S)$. For any realization τ of $\mathcal{T}$ and $t \geq 0$,*

$$\mathbb{E}_x[\hat{f}(X_t) \,|\, \mathcal{T}_t = \tau_t] \geq (T_t f(x))^{N_t^{\tau_t}},$$

where N_t^τ denotes the number of individuals at the top of τ_t.

P r o o f. We use induction over the number of edges of τ_t. If τ_t consists of a single edge, then $N_t^{\tau_t} = 1$ and

$$\mathbb{E}_x\left[\widehat{f}(X_t)\middle|\, \mathcal{T}_t = \tau_t\right] = \int f(y)\, q_t(x, dy) = (T_t f(x))^{N_t^{\tau_t}},$$

where $q_t(x, dy)$, $t \geq 0$, are the transition kernels of the particle migration process. If τ_t has two or more edges, let $t_1 < t$ denote the length of the edge containing the root, and let $\tau^{(1)}, \ldots, \tau^{(\beta)}$ be the subtrees of τ_t that stem from the first branching point of τ_t. Then we have $N_t^{\tau_t} = N_{t-t_1}^{\tau_{t-t_1}^{(1)}} + \cdots + N_{t-t_1}^{\tau_{t-t_1}^{(\beta)}}$, and, using the branching property, the induction hypothesis and Jensen's inequality,

$$\begin{aligned}
&\mathbb{E}_x\left[\widehat{f}(X_t)\middle|\, \mathcal{T}_t = \tau_t\right] \\
&= \int \mathbb{E}_z\left[\widehat{f}(X_{t-t_1})\middle|\, \mathcal{T}_{t-t_1} = \tau_{t-t_1}^{(1)}\right] \cdots \mathbb{E}_z\left[\widehat{f}(X_{t-t_1})\middle|\, \mathcal{T}_{t-t_1} = \tau_{t-t_1}^{(\beta)}\right] q_{t_1}(x, dz) \\
&\geq \int (T_{t-t_1} f(z))^{N_{t-t_1}^{\tau_{t-t_1}^{(1)}}} \cdots (T_{t-t_1} f(z))^{N_{t-t_1}^{\tau_{t-t_1}^{(\beta)}}} q_{t_1}(x, dz) \\
&= \int (T_{t-t_1} f(z))^{N_t^{\tau_t}} q_{t_1}(x, dz) \\
&\geq (T_t f(x))^{N_t^{\tau_t}}.
\end{aligned}$$

$\square$

Theorem 3. *Let $u_t(x)$, $t \geq 0$, $x \in S$ be the mild solution of the Fujita equation (10), with $u_0 \geq 0$ bounded and measurable. If for some $t > 0$ and $x \in S$,*

$$T_t u_0(x) \geq (Vt(\beta-1))^{-1/(\beta-1)},$$

then u blows up at x in finite time.

Proof. From Theorem 1 we know that $u_t(x) = \mathbb{E}_x[e^{S_t}\hat{f}(X_t)]$, where

$$\mathbb{E}_x[e^{S_t}\hat{f}(X_t)] \;=\; \mathbb{E}_x[\mathbb{E}_x[e^{S_t}\hat{f}(X_t)\,|\,\mathcal{T}_t]\,] \;=\; \mathbb{E}_x[e^{S_t}\mathbb{E}_x[\hat{f}(X_t)\,|\,\mathcal{T}_t]\,].$$

The proof is finished by applying the lemmas 1 and 2. □

6. Global solutions and finite-time blow up of systems of Fujita equations

In this section we consider systems of semilinear equations. For simplicity, we restrict ourselves to systems of the form

$$\begin{aligned} \frac{\partial u_t}{\partial t} &= A_1 u_t + V_1 u_t^{\beta_{11}} v_t^{\beta_{12}},\ t > 0, \\ \frac{\partial v_t}{\partial t} &= A_2 v_t + V_2 u_t^{\beta_{21}} v_t^{\beta_{22}},\ t > 0, \\ u_0 &= f,\ \ v_0 \;=\; g, \end{aligned} \tag{16}$$

where A_i is the generator of a strong Markov process in S with transition semigroup $\{T_t^i,\ t \geq 0\}$, $V_i > 0$ and $\beta_{ij} \in \{1, 2, \ldots\}$ are constants, $i, j = 1, 2$, and $f, g \in \mathcal{B}(S)$. The probabilistic representation of system (16), as well as its blowup properties, are derived in a manner similar to the univariate case, employing a multitype branching population instead of the monotype one that we used in preceding sections. To be precise, let us consider a population living in S, consisting of individuals of types 1 and 2. Any individual of type i lives an exponential lifetime of parameter V_i during which it develops a Markov motion with generator A_i. At the end of its life it branches, leaving an offspring constituted by β_{i1} individuals of type 1 and β_{i2} individuals of type 2. The newborns appear where the parent individual died and evolve independently in the same fashion.

We denote by X_t the configuration at time $t \geq 0$ of the two-type population described above. Note that X_t takes values in the space $\mathcal{N}_{\mathsf{f}}(S \times \{1, 2\})$ of finite counting measures on $S \times \{1, 2\}$, where the first component of a point $(x, i) \in S \times \{1, 2\}$ stands for the position, and the second component for the type of an individual $\delta_{(x,i)}$. Recall that

$$S_t \;=\; V_1 \int_0^t N_{s,1}\, ds + V_2 \int_0^t N_{s,2}\, ds,\ \ t \;\geq\; 0,$$

represents the (weighted) length of the ancestor's offspring tree, where $N_{t,i}$ is the number of individuals of type i in X_t. We define $N_t := N_{t,1} + N_{t,2}$, $t \geq 0$.

6.1. Existence of global solutions

Theorem 4. *Let (u_t, v_t) be the mild solution of System (16), and let $\varphi : S \times \{1,2\} \to \mathbb{R}_+$ be defined by $\varphi(x,1) = f(x)$, $\varphi(x,2) = g(x)$. Then (u_t, v_t) admits the representation*

$$(17)\quad u_t(x) = \mathbb{E}_{\delta_{(x,1)}}\left[e^{S_t}\hat{\varphi}(X_t)\right], \quad v_t(x) = \mathbb{E}_{\delta_{(x,2)}}\left[e^{S_t}\hat{\varphi}(X_t)\right], \quad t \geq 0,\ x \in S.$$

Moreover, if φ is bounded by 1, then the mild solution of (16) satisfies

$$\begin{aligned} u_t(x) &\leq T_t^1 f(x)\left(1 + \sum_{k=1}^{\infty} v_k(t)\right), \\ v_t(x) &\leq T_t^2 g(x)\left(1 + \sum_{k=1}^{\infty} v_k(t)\right), \end{aligned}$$

where

$$v_k(t) = \frac{\prod_{i=0}^{k-1}(1 + i(\beta^* - 1))}{k!}\left[V\int_0^t \left(\sup_{z\in S} T_s\varphi(z)\right)^{\beta_*-1} ds\right]^k,$$

with $V = V_1 \vee V_2$, $\beta_* = (\beta_{11}+\beta_{12}) \wedge (\beta_{21}+\beta_{22})$, and $\beta^* = (\beta_{11}+\beta_{12}) \vee (\beta_{21}+\beta_{22})$. In particular, if φ is bounded by 1 and satisfies

$$(18)\qquad (\mu^* - 1)V\int_0^\infty \left(\sup_{z\in S} T_s\varphi(z)\right)^{\mu_*-1} ds < 1,$$

then the corresponding mild solution of system (16) is global.

Proof. The proof of (17) is very similar to that of Theorem 1 and will not be given here. To prove the remaining assertions let us define the kernels $\mathbf{\Psi}_1$ and $\mathbf{\Psi}_2$ by

$$\begin{aligned} \int_{\mathcal{N}_{\mathfrak{f}}(S\times\{1,2\})} \mathbf{\Psi}_1(\mu, ds\, d\nu)v(\nu) &= (1 - \delta_{n0})V_1\, ds \sum_{i=1}^{n} T_s^1\left(\int v(\nu)\,\pi^{(\cdot,1)}(d\nu)\right)(x_i) \\ &\quad \times \prod_{\substack{l=1\\ l\neq i}}^{n} T_s^1 v(x_l) \prod_{h=1}^{m} T_s^2 v(y_h), \end{aligned}$$

$$\int_{\mathcal{N}_{\mathsf{f}}(S\times\{1,2\})} \mathbf{\Psi}_2(\mu, ds\, d\nu)v(\nu) \;=\; (1-\delta_{m0})V_2\, ds \sum_{j=1}^{m} T_s^2 \left(\int v(\nu)\, \pi^{(\cdot,2)}(d\nu) \right)(y_j) \times \prod_{l=1}^{n} T_s^1 v(x_l) \prod_{\substack{h=1 \\ h\neq j}}^{m} T_s^2 v(y_h)$$

for $\mu = \sum_{i=1}^n \delta_{(x_i,1)} + \sum_{j=1}^m \delta_{(y_j,2)} \in \mathcal{N}_{\mathsf{f}}(S\times\{1,2\})$, where $v : \mathcal{N}_{\mathsf{f}}(S\times\{1,2\}) \to \mathbb{R}_+$ is measurable and

$$\pi^{(z,i)}(d\nu) := \delta_{\beta_{i1}\delta_{(z,1)}+\beta_{i2}\delta_{(z,2)}}(d\nu), \quad (z,i) \in S\times\{1,2\}.$$

If $\mu = \sum_{i=1}^n \delta_{(x_i,1)} + \sum_{j=1}^m \delta_{(y_j,2)}$, then

$$\mathbb{E}_\mu \left[e^{S_t} \hat{\varphi}\,(X_t) \right] \;=\; \prod_{i=1}^n T_t^1 f(x_i) \prod_{j=1}^m T_t^2 g(y_j) + \int_0^t \int_{\mathcal{N}_{\mathsf{f}}(S\times\{1,2\})} \mathbf{\Psi}_1(\mu, ds\, d\nu) \mathbb{E}_\nu \left[e^{S_{t-s}} \hat{\varphi}\,(X_{t-s}) \right] + \int_0^t \int_{\mathcal{N}_{\mathsf{f}}(S\times\{1,2\})} \mathbf{\Psi}_2(\mu, ds\, d\nu) \mathbb{E}_\nu \left[e^{S_{t-s}} \hat{\varphi}\,(X_{t-s}) \right].$$

Hence, $\mathbb{E}_\mu \left[e^{S_t} \hat{\varphi}\,(X_t) \right]$ can be expanded as $\mathbb{E}_\mu \left[e^{S_t} \hat{\varphi}\,(X_t) \right] = \sum_{k=0}^\infty u_k(t,\mu)$, where $u_0(t,\mu) = \prod_{i=1}^n T_t^1 f(x_i) \prod_{j=1}^m T_t^2 g(y_j)$, and

$$u_{k+1}(t,\mu) \;=\; \int_0^t \int_{\mathcal{N}_{\mathsf{f}}(S\times\{1,2\})} \mathbf{\Psi}_1(\mu, ds\, d\nu) u_k(t-s,\nu) + \int_0^t \int_{\mathcal{N}_{\mathsf{f}}(S\times\{1,2\})} \mathbf{\Psi}_2(\mu, ds\, d\nu) u_k(t-s,\nu), \qquad k = 0,\, 1, \ldots .$$

Using induction one can prove that for $t \geq 0$, $\mu = \sum_{i=1}^n \delta_{(x_i,1)} + \sum_{j=1}^m \delta_{(y_j,2)}$ and $k = 0,\, 1, \ldots,$

$$u_k(t,\mu) \leq \frac{V^k}{k!} \prod_{i=0}^{k-1} (n+m+i(\beta^*-1)) \left[\int_0^t \left(\sup_{z\in S} T_s\varphi(z) \right)^{\beta_*-1} ds \right]^k \prod_{i=1}^n T_t^1 f(x_i) \prod_{j=1}^m T_t^2 g(y_j).$$

From here the proof proceeds as in the monotype case. □

6.2. A sufficient condition for blowup

Let $\mathbb{E}_{[n,m]}$ denote expectation when the initial two-type population consists of n type-1 and m type-2 individuals. We put $\beta_1 := \beta_{11} + \beta_{12}$, $\beta_2 := \beta_{21} + \beta_{22}$, and define

$$u_t^{[n,m]}(K) := \mathbb{E}_{[n,m]}[e^{S_t} K^{N_t}], \quad t \geq 0, \quad K \geq 0, \quad n,m \in \{0,1,\ldots\}.$$

Lemma 3. *For any $t \geq 0$, $K \geq 0$, and $n, m \in \{0, 1, \ldots\}$,*

$$u_t^{[n,m]}(K) \geq K^{n+m}\left[1 + \sum_{l=1}^{\infty} \frac{(V_* t)^l}{l!} \sum_{(\gamma_1,\ldots,\gamma_l)\in\{1,2\}^l} K^{\theta_l(\gamma_1,\ldots,\gamma_l)} \prod_{i=1}^{l}\left(\eta_{\gamma_i} + \sum_{j=1}^{i-1}(\beta_{\gamma_j\gamma_i} - \delta_{\gamma_j\gamma_i})\right)\right], \tag{19}$$

where $\theta_l(\gamma_1, \ldots, \gamma_l) = (l - \sum_{i=1}^{l}(\gamma_i - 1))(\beta_1 - 1) + \sum_{i=1}^{l}(\gamma_i - 1)(\beta_2 - 1)$, $\eta_1 = n$ *and* $\eta_2 = m$.

The proof of (19) follows closely the method of proof of Lemma 1, and will not be developed here. The details appear in [8].

Corollary 1. *Assume that* $2 \leq \beta_1 \leq \beta_2$.

(a) If $\beta_1 = \beta_2$ *or* $\beta_{11} \geq 2$, *then* $\mathbb{E}_{[1,0]}[e^{S_t} K^{N_t}] = \infty$ *for* $K \geq ct^{-\frac{1}{\beta_1 - 1}}$, $t > 0$.

(b) If $\beta_{11} = \beta_{22} = 0$, *then* $\mathbb{E}_{[1,0]}[e^{S_t} K^{N_t}] = \infty$ *for* $K \geq c' t^{-\frac{2}{\beta_{12}+\beta_{21}-2}}$, $t > 0$.

Here c *y* c' *are constants that may depend on* β_{ij} *and* $V_* := V_1 \wedge V_2$ *but not on* t.

P r o o f. If $\beta_1 = \beta_2$ consider a monotype population with exponential lifetimes of parameter V_* and branching numbers $\beta := \beta_1$. The first assertion in (a) then follows from the results in Section 5. For the proof of the second assertion in (a) we use Lemma 3. Indeed, it suffices to keep in the right side of (19) only those terms $K^{\theta_l(\gamma_1,\ldots,\gamma_l)} \prod_{i=1}^{l}(\eta_{\gamma_i} + \sum_{j=1}^{i-1}(\beta_{\gamma_j\gamma_i} - \delta_{\gamma_j\gamma_i}))$ for which $(\gamma_1, \ldots, \gamma_l)$ is of the form $(1, \ldots, 1)$. Lemma 3 renders

$$u_t^{[n,m]}(K) \geq K^{n+m}\left[1 + \sum_{l=1}^{\infty}\prod_{i=1}^{l}(n + (i-1)(\beta_{11} - 1))\frac{(V_* t K^{\beta_1 - 1})^l}{l!}\right].$$

Hence, if $t > 0$ and $K \geq (V_* t(\beta_{11} - 1))^{-\frac{1}{(\beta_1 - 1)}}$, then $(V_* t K^{\beta_1 - 1})^l \geq \frac{1}{(\beta_{11}-1)^l}$ and therefore

$$\prod_{i=1}^{l}(n + (i-1)(\beta_{11} - 1))\frac{(V_* t K^{\beta_1 - 1})^l}{l!} \geq \frac{1}{l!}\prod_{i=1}^{l}\left(\frac{n}{\beta_{11} - 1} + i - 1\right) \geq \frac{n}{\beta_{11} - 1}\frac{1}{l}.$$

It follows that for any $t > 0$ and $K \geq (V_* t(\beta_{11} - 1))^{-\frac{1}{\beta_1 - 1}}$, $u_t^{[n,m]}(K) = \infty$ for all $n \geq 1$ and $m \geq 0$. The proof of (b) is similar. □

The extension of Theorem 3 to systems of equations is more delicate. We are not aware of a multivariate version of Lemma 2 in the generality of our setting. For this reason, in the remaining of this section we restrict ourselves to the particular case $S = \mathbb{R}^d$. Moreover, we assume that the motion process of particles of type i has transition densities $\{q_t^i(x,y),\ t>0\}$, where $q_t^i(x,y) = q_t^i(x-y)$ and $q_t^i(\cdot)$ is symmetric unimodal, $i = 1, 2$. Spherically symmetric stable processes, and continuous-time random walks with symmetric unimodal jump distributions meet these assumptions.

Suppose $\{X_t,\ t \geq 0\}$ starts with an ancestor δ_z and let $\mathcal{T} = \{\mathcal{T}_t,\ t \geq 0\}$ denote its offspring tree. For each fixed realization τ of $\mathcal{T}$ let us denote by $\partial\tau_t$ the set of branches of τ_t, where by a branch we understand a set of edges leading from the root to an individual in the top of τ_t. Notice that the edges of τ are of types 1 and 2; we denote by $\overline{e}$ the type of edge e. For any branch $b_t \in \partial\tau_t$ let $\{W_s^{x,b_t},\ 0 \leq s \leq t\}$ be the process starting in $x \in \mathbb{R}^d$ which follows the motion with generator $A_{\overline{e}}$ along the edge e of b_t, where $W_\cdot^{x,b_t}$, $b_t \in \partial\tau_t$, are assumed to be independent. We denote by $\{X_t^{\tau,x},\ t \geq 0\}$ the branching population in $\mathbb{R}^d$ indexed by τ, starting with an ancestor at position $x \in \mathbb{R}^d$, and whose individuals follow the motion of generator $A_{\overline{e}}$ along the edge e of τ. Using induction on the number of edges of τ and our assumptions on $\{q_t^i(x-y),\ t>0\}$, one can show that for any symmetric and unimodal $f \in \mathcal{B}(\mathbb{R}^d)$, the function

$$x \to \mathbb{E}[\hat{f}(X_t^{\tau,x})], \quad x \in \mathbb{R}^d,$$

is symmetric and unimodal as well. Since the convolution of symmetric unimodal functions is again symmetric and unimodal (see [10], page 98), similarly as in the proof of Lemma 2 it follows that

Lemma 4. *For any symmetric unimodal $f \in \mathcal{B}(\mathbb{R}^d)$ and any realization τ of $\mathcal{T}$,*

$$\mathbb{E}\left[\widehat{f}\,(X_t^{\tau,x})\right] \geq \prod_{b_t \in \partial\tau_t} \mathbb{E}\left[f\left(W_t^{x,b_t}\right)\right], \quad x \in \mathbb{R}^d, \quad t \geq 0.$$

Therefore $\mathbb{E}_{[1,0]}\Big[e^{S_t}\widehat{f}\left(X_t^{\mathcal{T},x}\right)\Big] = \mathbb{E}_{[1,0]}\Big[e^{S_t}\mathbb{E}_{[1,0]}\Big[\widehat{f}\left(X_t^{\mathcal{T},x}\right)\Big|\,\mathcal{T}\Big]\Big] \geq \mathbb{E}_{[1,0]}\big[e^{S_t}K^{N_t}\big]$. This renders the following result.

Proposition 1. *Let $\{W_t^1,\ t \geq 0\}$ and $\{W_t^2,\ t \geq 0\}$ be independent processes in $\mathbb{R}^d$ with generators A_1 and A_2 respectively, both initiating in the origin. Let $f \in \mathcal{B}(\mathbb{R}^d)$ be symmetric and unimodal. If for some $x \in \mathbb{R}^d$ and $t > 0$ the number*

$$K := \inf_{0 \leq r \leq t} \mathbb{E}\left[f(x + W_r^1 + W_{t-r}^2)\right]$$

satisfies the conditions of Corollary 1 (a) or (b), then $\mathbb{E}_{[1,0]}\left[e^{S_t}\widehat{f}\left(X_t^{\mathcal{T},x}\right)\right] = \infty$.

A criterion for blowup of System (16), similar to Theorem 3, can be proved provided the initial values in (16) satisfy $f \wedge g \geq h$, where $h \not\equiv 0$ is nonnegative, symmetric and unimodal. Alternatively, if in addition to our assumptions on the transition densities $\{q_t^i(x-y),\ t>0\}$ we suppose that $q_t^i(\cdot)$ is strictly positive and continuous for each $t>0$, $i=1,2$, then there exist $t_0>0$ such that $f\wedge g$ is bounded from below on the unit ball by a positive constant k. Restarting the system at time t_0 if necessary, we can assume $f \wedge f \geq k1_{B_1(0)} := h$, where $B_1(0) \subset \mathbb{R}^d$ denotes the unit ball centered at the origin. This, combined with Proposition 1 proves the following theorem.

Theorem 5. *Suppose that for any ball $B \subset \mathbb{R}^d$ centered at the origin, the number $K := \inf_{0\leq r\leq t} \mathbb{P}\left[W_r^1 + W_{t-r}^2 \in B\right]$ satisfies the conditions of Corollary 1 (a) or (b), where $\{W_t^1,\ t\geq 0\}$ and $\{W_t^2,\ t\geq 0\}$ denote independent processes in $\mathbb{R}^d$ with generators A_1 and A_2 respectively, with $W_0^1 = W_0^2 = 0$. If the transition densities $\{q_t^i,\ t\geq 0\}$ of $\{W_t^i,\ t\geq 0\}$ satisfy*

(a) $q_t^i(x,y) = q_t^i(x-y)$, $x,y\in\mathbb{R}^d$, and $q_t^i(\cdot)$ is symmetric unimodal,

(b) $q_t^i(\cdot)$ is contiuous and satisfies $q_t^i(x)>0$, $x\in\mathbb{R}^d$

for $i=1,2$ and all $t>0$, then the mild solution of System (16) blows up in finite time for all initial values (f,g) satisfying $f(x) \geq k_1\, 1_B(x)$, $g(x) \geq k_2\, 1_{B'}(x)$, $x\in\mathbb{R}^d$, for some constants $k_1, k_2 > 0$ and balls $B, B' \subset \mathbb{R}^d$.

7. Blow up of the Dirichlet boundary value problem

In this section we are interested in blowup of mild solutions of the Dirichlet boundary value problem

$$(20) \qquad \frac{\partial u}{\partial t} = \Delta u + V u^\beta,\ t>0, \qquad u(0,x) = f(x),\ x\in G, \qquad u|_{\partial G} \equiv 0,$$

where $G\subset\mathbb{R}^d$ is a bounded domain. Let $B\equiv(B_t)$ be the Brownian motion in $\mathbb{R}^d$ with variance parameter two, and let $\{T_t,\ t\geq 0\}$ denote the strongly continuous semigroup in $L_2(G)$ corresponding to the process B killed at $\tau := \inf\{t>0 | B_t \in \partial G\}$. Suppose that G is regular in the sense that B hits the complement of G immediately after time zero when started from any point in ∂G. Then one can show that the semigroup $\{T_t,\ t\geq 0\}$ is strongly continuous in the space $C_0(G)$ of continuous functions on G vanishing at ∂G.

Let $\{\varphi_n\}_{n=0}^{\infty} \subset C_0(G)$ and $0 < \lambda_0 < \lambda_1 \leq \cdots$ be the nontrivial solutions of the eigenvalue problem

$$\Delta\varphi(x) + \lambda\varphi(x) = 0,\ x \in G, \qquad \varphi(x) = 0,\ x \in \partial G,$$

where φ_n is normalized by $\|\varphi_n\|_2 = 1$ (here $\|\cdot\|_p$ stands for the norm in L_p). It is well known that the eigenvalue λ_0 has multiplicity one and that the function φ_0 is strictly positive on G. Moreover, for any $t > 0$ and any $f \in C_0(G)$,

$$T_t f(x) = \sum_{n=0}^{\infty} e^{-\lambda_n t}\varphi_n(x) \int_G f(y)\varphi_n(y)\,dy,\ t > 0,\ \ x \in G. \tag{21}$$

We say that (T_t) is intrinsically ultracontractive (IUC) provided that for all $t > 0$ there exists a positive constant c_t such that

$$|T_t f(x)| \leq c_t \|f\|_2 \varphi_0(x),\ x \in G, f \in C_0(G). \tag{22}$$

The following assertions have been proved in [2].

Proposition 2. *If G obeys both an exterior and an interior cone conditions, then G is regular and (T_t) is IUC.*

In [1] it is proved that the IUC property holds for a large class of domains G. The following theorem, where $V > 0$ and $\beta > 1$ are constants, is proved in [7].

Theorem 6. *Assume that $\{T_t,\ t \geq 0\}$ is IUC and let $f \in C_0(G)$ be nonnegative. If*

$$\langle f, \varphi_0 \rangle_{L_2} > \left(\frac{\lambda_0}{V}\right)^{1/(\beta-1)} \|\varphi_0\|_1, \tag{23}$$

then the mild solution $u(t,x)$ of (20) blows up in finite time.

A probabilistic interpretation of blowup of mild solutions of (20) is as follows. Let

$$Q_t g(x) = e^{\lambda_0 t}\varphi_0^{-1}(x) T_t(g\varphi_0)(x),\ x \in G,\ g \in C_b(G).$$

Then $\{Q_t,\ t \geq 0\}$ is a strongly continuous contraction semigroup on $C_b(G)$ having $\varphi_0^2(x)\,dx$ as its (unique) invariant measure. In fact, for any $g \in C_b(G)$ and $f \equiv g\varphi_0$,

$$\sup_{x \in G} |Q_t g(x) - g(x)| \leq \sum_{n=1}^{\infty} \left(1 - e^{-(\lambda_n - \lambda_0)t}\right) \sup_{x \in G} \left|\frac{\varphi_n(x)}{\varphi_0(x)}\right| |\langle f, \varphi_n \rangle|$$

by (21), and the series in the last inequality goes to 0 as $t \downarrow 0$ due to IUC of $\{T_t,\ t \geq 0\}$. The generator H of $\{Q_t,\ t \geq 0\}$ is given by

$$Hg = \varphi_0^{-1}(\Delta^G + \lambda_0)(g\varphi_0),\ g \in \mathrm{Dom}(H) := \{g \in C_b(G) : g\varphi_0 \in \mathrm{Dom}(\Delta^G)\},$$

where Δ^G is the generator of the killed Brownian motion. From the self-adjointness of Δ^G it follows that $\int Hg(x)\varphi(x)_0^2\,dx = 0$ for each $g \in \mathrm{Dom}(H)$, which yields the Q_t-invariance of $\varphi^2(x)\,dx$. Writing $\mathbb{E}[g] := \int g(x)\varphi_0^2(x)\,dx$, we conclude that $\mathbb{E}[Q_t g] = \mathbb{E}[g]$, $t \geq 0$, $g \in C_b(G)$.

We define

$$w(t,x) = e^{\lambda_0 t}\frac{u(t,x)}{\varphi_0(x)} \text{ and } z(t,x) = e^{-\lambda_0 t}\varphi_0(x),\ x \in G,\ t \geq 0, \tag{24}$$

where

$$u(t) = T_t f + V\int_0^t T_s u(t-s)^\beta\,ds,\ t \geq 0, \tag{25}$$

is the mild solution of (20). Multiplying both sides of (25) by $\varphi_0^{-1}(x)e^{\lambda_0 t}$ yields

$$w(t,x) = Q_t g(x) + V\int_0^t Q_s w(t-s,\cdot)^\beta z(t-s,\cdot)^{\beta-1}(x)\,ds,\ x \in G,\ t \geq 0. \tag{26}$$

If $\beta > 1$ is an integer it is possible to represent the solution of Equation (26) as an expectation functional of a related branching particle system, similarly as we did in theorems 1 and 4. Indeed, consider a two-type population in G with the individuals evolving independently in the following way: a particle of type 1 lives an exponential lifetime of mean $1/V$ during which it moves according to the semigroup $\{Q_t,\ t \geq 0\}$. At the end of its life it branches producing β individuals of type 1 and $\beta - 1$ individuals of type 2, all appearing at the mother's death position. The particles of type 2 develop independent killed Brownian motions and do not branch. For $i = 1, 2$, let $X_{t,i}^x$ denote the random finite point measure on G representing the population of type-i individuals present at time $t \geq 0$, starting with an ancestor of type 1 at position $x \in G$. Then the solution $w(t,x)$ of (26) is given by

$$w(t,x) = \mathbb{E}\left[e^{S_t^x}\prod_{z\in\mathrm{supp}(X_{t,1}^x)} g(z)\prod_{z\in\mathrm{supp}(X_{t,2}^x)}\varphi_0(z)\right],\quad t \geq 0,\quad x \in G, \tag{27}$$

where $V^{-1}S_t^x =: \int_0^t\int X_{s,i}^x(dy)\,ds$ represents the total time length of the family tree of type 1 up to time t. Since $\varphi_0^2(x)\,dx$ is the invariant measure of $\{Q_t,\ t \geq 0\}$, if φ_0 decays sufficiently fast near ∂G and the points $x \in G$ for which $g(x) \equiv f(x)/\varphi_0(x)$ is large lie on regions where $\varphi_0^2(x)\,dx$ puts little mass (equivalently, if $\langle f, \varphi_0 \rangle$ is small), then the decay of $\prod_{z \in \operatorname{supp} X^x_{t,2}} \varphi_0(z)$ when $t \to \infty$ is able to counteract the contribution of the factor $e^{S^x_t} \prod_{z \in \operatorname{supp} X^x_{t,1}} g(z)$ to the expectation in (27), thus preventing blowup of $w(t)$ and hence of $u(t)$.

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José Alfredo López-Mimbela
Centro de Investigación en Matemáticas
Apartado Postal 402
36000 Guanajuato, Mexico
e-mail: `jalfredo@cimat.mx`